Diffusion and Self–Organized Criticality in Ricci Flow Evolution of Einstein and Finsler Spaces

Sergiu I. Vacaru[*]

Science Department, University "Al. I. Cuza" Iaşi,
54 Lascar Catargi street, 700107, Iaşi, Romania

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Abstract

Imposing non–integrable constraints on Ricci flows of (pseudo) Riemannian metrics we model mutual transforms to, and from, non–Riemannian spaces. Such evolutions of geometries and physical theories can be modelled for nonholonomic manifolds and vector/ tangent bundles enabled with fundamental geometric objects determining Lagrange–Finsler and/or Einstein spaces. Prescribing corresponding classes of generating functions, we construct different types of stochastic, fractional, nonholonomic etc models of evolution for nonlinear dynamical systems, exact solutions of Einstein equations and/or Lagrange–Finsler configurations. The main result of this paper consists in a proof of existence of unique and positive solutions of nonlinear diffusion equations which can be related to stochastic solutions in gravity and Ricci flow theory. This allows us to formulate stochastic modifications of Perelman's functionals and prove the main theorems for stochastic Ricci flow evolution. We show that nonholonomic Ricci flow diffusion can be with self–organized critical behavior, for gravitational and Lagrange–Finsler systems, and that a statistical/ thermodynamic analogy to stochastic geometric evolution can be formulated.

Keywords: stochastic process, nonlinear diffusion, Ricci flows, Einstein manifold, Finsler space, nonlinear connection, nonholonomic manifold, exact solutions in gravity, self–organized criticality.

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[*]sergiu.vacaru@uaic.ro, Sergiu.Vacaru@gmail.com;
http://www.scribd.com/people/view/1455460-sergiu

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Contents

1   Introduction

The purpose of this paper is to analyze models of Ricci flows of random metrics with self–organize criticality in general relativity (and various modifications, for instance, of Lagrange–Finsler type theories of gravity) in the framework of stochastic evolution equations. In other words, we shall analyze scenarios of stochastic/diffusion Ricci evolution of curved spaces with rich geometric/physical structure[1] resulting in random gravitational systems with critical points as attractors.

This is a partner work to [1]; our approach to stochastic solutions and diffusion of gravitational fields is elaborated in the framework of the theory of stochastic evolution equations following the methods elaborated in [2, 3, 4, 5] (see also references therein) but modified/adapted for curved spaces and nonholonomic Ricci evolution of Riemannian and non–Riemannian geometries, [6, 7, 8, 9]).

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[1]when, for instance, metrics are solutions of Einstein equations with certain prescribed symmetries and/or nonholonomic constraints, solitonic configurations, stochastic properties etc

The theory of stochastic processes and diffusion on curved spaces has been studied in mathematics and physics from different perspectives related to new directions in differential geometry and partial differential equations, geometric analysis and evolution theory, kinetic and thermodynamic processes with local anisotropy, and various applications in cosmology and astrophysics. We refer to a series of works containing original key ideas, methods and reviews of results [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

In our recent paper [1], we studied exact solutions in gravity defined by stochastic generating functions and nonholonomic diffusion of gravitational fields. Such generic off–diagonal metrics (which can not be diagonalized by coordinate transforms) describe generic nonlinear and random processes with very complex structure of gravitational "spacetime ether", for instance, solitonic interactions [22, 23], fractional configurations [24, 25, 26] and nonlinear wave interactions and geometric evolution [27, 28, 29].

In this work, the gravitational stochastic Ricci flow and field interactions phenomena with self–organized criticality are studied following a synthesis of the method of anholonomic deformation/frames of constructing exact solutions in gravity [30, 31, 32, 33, 34] and the theory of nonholonomic diffusion on curved spaces (additionally to above mentioned works, we cite our contributions on nonholonomic diffusion in (super) vector bundles and locally anisotropic kinetics and thermodynamics [13, 35, 14, 15]). To the best our knowledge this is the first attempt to provide an unified geometric formalism for locally anisotropic diffusion processes and Ricci flows of Einstein/–Finsler spaces, when certain classes of nonholonomic constraints result in stochastic evolution equations which can be approached in a mathematically strict way [2, 3, 4, 5].[2]

This paper is organized as follow. In section 2 we provide an introduction to the theory of Ricci flow evolution of nonholonomic geometries. Section

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[2]During last two decades, self–organized criticality is widely studied in physics [36, 37, 38, 40, 41]. Our idea is that such effects are possible for gravitational fields and their evolution being derived as corresponding solutions. Classical vacuum structure in general relativity may have various sophisticate nontrivial topological and geometric configurations with possible stochastic, solitonic, instanton, black hole, wormhole etc properties. We can say that a curved spacetime is modeled by a nontrivial fundamental gravitational "ether" which, as a matter of principe, may have a very complex nonlinear sure and/or random behavior which may result in configurations with self–organized criticality. The term "porous" media is a conventional one for gravitational configurations related to corresponding equations which are formally similar to certain equations for real porous matter. In this work, we show in explicit form what type of nonholonomic constraints we have to impose on gravitational field equations, and their possible geometric evolution, in order to have critical points as attractors.

3 is a development of some recent results on stochastic diffusion equations (in our case) on nonholonomic manifolds and to proof of existence of unique and positive solutions for such systems in gravity and geometric mechanics. In section 4 we develop the theory of stochastic nonholonomic Ricci flows: there are considered stochastic modifications of Perelman's functionals and proven the main theorems for stochastic evolution equations. Finally, a statistical analogy for stochastic Ricci flows is proposed.

2 Ricci Flow Evolution and Nonholonomic Manifolds

The Ricci flow theory [42, 43, 44, 45], related geometric analysis and various applications (see [46, 47, 48] for reviews of results and methods) became one of the most intensively developing branch of modern mathematics. The most important achievement of this theory was the proof of W. Thurston's Geometrization Conjecture by Grisha Perelman [43, 44, 45]. The main results on Ricci flow evolution were proved originally for (pseudo) Riemannian and Kähler geometries. In a series of our works [6, 7, 8, 9, 24, 49, 50], we studied Ricci flow evolutions of geometries and physical models (of gravity with symmetric and nonsymmetric metrics, geometric mechanics, fractional and noncommutative generalizations) when the field equations are subjected to nonholonomic constraints and the field/evolution solutions, mutually transform as Riemann and generalized non–Riemann geometries. In this section we outline the anholonomic deformation method of constructing exact solutions in gravity and Ricci flow theory in a form necessary for further developments (in next sections) for stochastic evolution and Ricci flow diffusion.

2.1 Modelling Einstein and Finsler geometries on nonholonomic manifolds

In the following we shall consider a nonholonomic manifold $\mathbf{V} = (V, \mathcal{N})$ enabled with nonlinear connection (N–connection) structure $\mathbf{N} : T\mathbf{V} \rightarrow h\mathbf{V} \oplus v\mathbf{V}$ as a nonholonomic distribution $\mathcal{N} = \mathbf{N}$ defining a conventional horizontal (h) and vertical (v) splitting on V. We consider that V is a four dimensional, 4–d, (pseudo) Riemann manifold of signature $(+, +, -+)$. For models of Lagrange–Finsler geometry, we can take $\mathbf{V} = TM$ to be the total

space of a manifold M, when the typical fiber in such a manifold is provided with a hyperbolic structure in order to mimic a local (pseudo) Euclidian structure. We shall use the notations from [1] (readers may find details on the geometry of nonholonomic manifolds [51, 52, 53]) in monograph [54]; for purposes of modern geometry, mechanics and physics and mathematical relativity, the formalism is developed in [56, 32, 30]; the concept of N–connection is contained in coefficient form in [55] being developed in details on a number of works on Lagrange–Finsler geometry and generalizations/ modifications [57, 58, 35, 7, 9]).

We label local coordinates on $\mathbf{V}$ in the form $u^\alpha = (x^i, y^a)$, were $i, j, ... = 1, 2$ and $a, b, ... = 3, 4$ (in brief, we write $\mathbf{u} = (\mathbf{x}, \mathbf{y})$); similar values are taken by any variants of primed, underlined etc indices, for instance, $\underline{\alpha} = (\underline{i}, \underline{a})$ and $\beta' = (j', b')$. A local frame and co–frame are written $e_\alpha = (e_i, e_a)$ and $e^\beta = (e^j, e^b)$, when frame transforms are parametrized $e_\alpha = e_\alpha^{\alpha'}(\mathbf{u})e_{\alpha'}$.

2.1.1 Nonlinear connections and Einstein equations

A N–connection structure on $\mathbf{V}$, with local coefficients $\{N_i^a\}$ stated with respect to a coordinate basis, allows us to define the so–called N–elongated (equivalently, N–adapted) frames, i.e. , respectively, partial derivatives and differentials,

$$\mathbf{e}_\alpha \doteqdot \left(\mathbf{e}_i = \partial_i - N_i^a \partial_a, e_b = \partial_b = \frac{\partial}{\partial y^b}\right), \tag{1}$$

$$\mathbf{e}^\beta \doteqdot \left(e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i\right). \tag{2}$$

With respect to such bases, the geometric objects are written in N–adapted form and called as distinguished objects (in brief, d–objects), for instance, d–vectors, d–tensors, d–connections etc.

Our geometric arena consists from nonholonomic manifolds/ bundles (we shall use as equivalent the terms, spaces, or spacetimes, for corresponding signatures) given by data $(\mathbf{N}, \mathbf{g}, \mathbf{D})$, where the d–metric $\mathbf{g}$ is parametrized in the form

$$\mathbf{g} = g_{ij} dx^i \otimes dx^j + h_{ab}(dy^a + N_k^a dx^k) \otimes (dy^b + N_k^b dx^k), \tag{3}$$

and $\mathbf{D}$ may be taken to be the canonical d–connection $\widehat{\mathbf{D}} = \left(h\widehat{D}, v\widehat{D}\right)$ uniquely defined from the conditions that it is metric compatible, $\widehat{\mathbf{D}}\mathbf{g} = 0$, and with vanishing "pure" h– and v–components of torsion $\widehat{\mathbf{T}}$ of $\widehat{\mathbf{D}}$. Even there are nontrivial h–v components of torsion, there is a distortion relation

$$\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}, \tag{4}$$

where all values, i.e. the canonical d–connection, $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^\alpha{}_{\beta\gamma}\}$, the Levi–Civita connection $\nabla = \{\Gamma^\alpha{}_{\beta\gamma}\}$ (subjected to the conditions that it is torsionless and $\nabla\mathbf{g} = 0$) and the distortion, $\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^\alpha{}_{\beta\gamma} \sim \widehat{\mathbf{T}}^\alpha{}_{\beta\gamma}\}$, are completely and uniquely defined by the same metric tensor $\mathbf{g}$. We can work equivalently with both linear connections $\widehat{\mathbf{D}}$ and/or ∇ (the first one is N–adapted but the second one is not).[3]

The Einstein equations in general relativity can be written for $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\,{}^sR = \mathbf{\Upsilon}_{\beta\delta}, \tag{5}$$

$$\widehat{L}^c_{aj} = e_a(N^c_j),\ \widehat{C}^i_{jb} = 0,\ \Omega^a{}_{ji} = 0, \tag{6}$$

where $\widehat{\mathbf{R}}_{\beta\delta}$ is the Ricci tensor for $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta}$, ${}^sR = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\beta\delta}$ and the source d–tensor $\mathbf{\Upsilon}_{\beta\delta}$ is constructed for the same metric but with $\widehat{\mathbf{D}}$ (formulas are similar to those in general relativity with ∇; we consider $\mathbf{\Upsilon}_{\beta\delta} \to \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$). If the constraints (6) are satisfied the distortion d–tensors $Z^\gamma{}_{\alpha\beta}$ from (4), determined by d–torsion $\widehat{\mathbf{T}}^\gamma{}_{\alpha\beta}$, became zero and (5) are equivalent to the "standard" equations

$$R_{\beta\delta} - \frac{1}{2}g_{\beta\delta}R = \varkappa T_{\beta\delta}, \tag{7}$$

written for the Levi–Civita connection $\nabla = \{\Gamma^\gamma{}_{\alpha\beta}\}$. In formulas (7), $R_{\beta\delta}$ and R are respectively the Ricci tensor and scalar curvature of ∇; it is also considered the energy–momentum tensor for matter, $T_{\alpha\beta}$, where $\varkappa = const$.

Here we note that it is not possible to integrate analytically, in general form, the Levi–Civita form of the system of partial differential equations (7) because of it generic nonlinearity and complexity. Nevertheless, it is possible to solve in very general forms the version for the canonical d–connection (and for the Cartan d–connection in Finsler gravity) the system (5), see details in Refs. [33, 34, 31], which motivates the idea to introduce the so–called Lagrange–Finsler variables in general relativity. Imposing additionally the conditions (6), we can restrict the nonholonomic integral varieties and generate solution of Einstein equations for the Levi–Civita connection.

2.1.2 Lagrange–Finsler structures and Einstein gravity

Let us consider how data $(\mathbf{N}, \mathbf{g}, \mathbf{D})$ can be parametrized on $\mathbf{V} = TM$ in such a form that they will define a Lagrange and/or Finsler geometry.

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[3]The formulas for N–adapted coefficients of geometric objects (4) are given in many our works (see, for instance, [1, 32, 30] and [56], for explicit constructions in Lagrange–Finsler gravity). For simplicity, we omit such considerations in this work.

We introduce frame transforms $^{L}g_{\alpha'\beta'}(\mathbf{u}) = e^{\alpha}_{\ \alpha'}(\mathbf{u})e^{\beta}_{\ \beta'}(\mathbf{u})\mathbf{g}_{\alpha\beta}(\mathbf{u})$, when the coefficients of d–metric (3) are transformed into coefficients of

$$^{L}\mathbf{g} = \ ^{L}g_{ij}dx^i \otimes dx^j + \ ^{L}h_{ab}(dy^a + \ ^{L}N^a_k dx^k)\otimes(dy^b + \ ^{L}N^b_k dx^k), \quad (8)$$

where $^{L}g_{ij} \sim \ ^{L}h_{ab} = \frac{1}{2}\frac{\partial^2 L(x^i,y^c)}{\partial y^a \partial y^b}$ and $^{L}N^a_k$ are respectively the Hessian and canonical N–connection defined by a regular Lagrangian $L(\mathbf{u}) = L(x^i, y^c)$, see details in [56, 32, 30] (and, for Ricci flows of Lagrange–Finsler structures, [8]). The d–metric (8) is called the Sasaki lift on TM of a regular Lagrange structure. In arbitrary frames/coordinates, a Lagrange d–metric $^{L}\mathbf{g}$ is represented in the form $\mathbf{g}$ (3) and, inversely, any $\mathbf{g}$ can be transformed into a $^{L}\mathbf{g}$, into certain Lagrange variables for a correspondingly chosen distribution/generating Lagrange function $L(\mathbf{u})$. The formal constructions can be performed geometrically in similar forms on (pseudo) Riemannian space-times and tangent bundles but have different physical interpretations (in the first case, we can elaborate analogous Lagrange–Finsler models of Einstein gravity but in the second case it is provided a Finsler similar geometrization of regular Lagrange mechanics).

We note that taking $L = F^2(x, y)$, where the homogeneous on y–variables real function F is the fundamental/generating Finsler function (see rigorous mathematical definitions and details in [56, 57, 58]), we model a Finsler geometry on a (pseudo) Riemannian manifold or on $\mathbf{V} = TM$. Using the geometric formalism of nonholonomic distributions, associated N–connections and the geometry of nonholonomic manifolds, we work in a unified form with all types of (pseudo) Riemann and Lagrange–Finsler geometries. We distinguish such constructions by additional suppositions on the structure of manifold/bundle spaces and the type of linear connection, $\widehat{\mathbf{D}}, \nabla$ or (the Cartan–Finsler d–connection) $^{c}\mathbf{D}$, we chose for the geometric data $(\mathbf{N}, \mathbf{g}, \mathbf{D})$.

Finally, we shall say that there are used nonholonomic (Lagrange, or Finsler, variables) in Einstein gravity on $\mathbf{V}$ if the theory is defined by any data $\left(\mathbf{N} \sim \ ^{L}\mathbf{N}; \mathbf{g} \sim \ ^{L}\mathbf{g}; \mathbf{D} \sim \widehat{\mathbf{D}}, \text{ or } \ ^{c}\mathbf{D}\right)$, which can be equivalently transformed, via distortion relations of type (4), into data $(\mathbf{g}, \nabla)$. For theories on TM, we can elaborate Einstein–Lagrange/–Finsler like theories (with metrics and connections depending on velocity type coordinates, y^a) using instead of ∇ any convenient for physical purposes $\widehat{\mathbf{D}}$, or $^{c}\mathbf{D}$ (or any other metric compatible d–connection completely defined by $\mathbf{g}$ and adapted to a chosen $\mathbf{N}$). In abstract form, a N–adapted Einstein model on a general nonholonomic manifold $\mathbf{V}$ generates for corresponding nonholonomic distributions an Einstein and/or Lagrange–Finsler, spacetime geometry.

2.2 Nonholonomic Ricci flows of Einstein/–Finsler spaces

The original Ricci flow theory was proposed [42] with an evolution equation for a set of Riemannian metrics $g_{\underline{\alpha}\underline{\beta}}(\chi)$ and corresponding Ricci tensors $R_{\underline{\alpha}\underline{\beta}}(\chi)$ parametrized by a real parameter χ. We can write the Hamilton's equations in the so–called normalized form, see details in [46, 47, 48],

$$\frac{\partial}{\partial \chi} g_{\underline{\alpha}\underline{\beta}} = -2\ R_{\underline{\alpha}\underline{\beta}} + \frac{2r}{5} R g_{\underline{\alpha}\underline{\beta}}, \tag{9}$$

describing (holonomic) Ricci flows with respect to a coordinate base $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$; the normalizing factor $r = \int\ R dVol/Vol$ is introduced in order to preserve the volume Vol; $R_{\underline{\alpha}\underline{\beta}}$ and $R = g^{\underline{\alpha}\underline{\beta}}\ R_{\underline{\alpha}\underline{\beta}}$ are computed for the Levi–Civita connection ∇.[4] Grisha Perelman's fundamental results [43, 44, 45] were based on original idea to prove that the Ricci flow is not only a gradient flow but also can be defined as a dynamical system on the spaces of Riemannian metrics. He introduced two Lyapunov type functionals and proved that evolution equations of type (9) can be derived following a corresponding variational calculus.

In our approach, we studied modified Ricci flow evolution equations, and the corresponding N–adapted functionals, for generalized commutative and noncommutative geometries (Riemann and Lagrange–Finsler ones) [6, 7, 8, 9, 24]. We also were interested to study Ricci flows of exact solutions in various theories of gravity [49, 50, 28, 29].   In this work, we shall elaborate a theory of Ricci evolution with self–organized criticality for stochastic solutions for Einstein spaces following geometric using the canonical d–connection $\widehat{\mathbf{D}}$.

If $\nabla \to \widehat{\mathbf{D}}$, we have to change $R_{\alpha\beta} \to \widehat{\mathbf{R}}_{\alpha\beta}$ which transforms equations (9) into an N–adapted system of evolution equations for Ricci flows of (in this work, we consider symmetric metrics, see details in [6, 7]),

$$\frac{\partial}{\partial \chi} g_{ij} = 2\left[N_i^a N_j^b \left(\widehat{\underline{R}}_{ab} - \lambda h_{ab} \right) - \widehat{\underline{R}}_{ij} + \lambda g_{ij} \right] - h_{cd} \frac{\partial}{\partial \chi} (N_i^c N_j^d), \tag{10}$$

$$\frac{\partial}{\partial \chi} h_{ab} = -2\left(\widehat{\underline{R}}_{ab} - \lambda h_{ab} \right), \tag{11}$$

$$\widehat{R}_{ia} = 0 \text{ and } \widehat{R}_{ai} = 0, \tag{12}$$

where the Ricci coefficients $\widehat{\underline{R}}_{ij}$ and $\widehat{\underline{R}}_{ab}$ are computed with respect to coordinate coframes and the cosmological constant λ includes the normalization

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[4]We underline the indices with respect to the coordinate bases but not with respect to some 'N–elongated' local bases.

factor $\frac{2r}{5}$. On nonholonomic manifolds, we can prescribe any convenient for our purposes nonholonomic distributions. For simplicity, we shall fix such distributions when $\frac{\partial}{\partial\chi}(N_i^c) = 0$ and model stochastic and/or Ricci flow evolution for nonholonomic Einstein manifolds with $\widehat{R}_{ab} = \lambda h_{ab}$ and $\widehat{R}_{ij} = \lambda g_{ij}$ for so–called "stationary configurations", when the equations (5) with the source determined by λ are satisfied. The h– and v–components of Ricci flow evolutions of gravitational configurations for $\frac{\partial}{\partial\chi}(N_i^c) = 0$, are similar to those derived from Perelman's nonholonomic functionals after corresponding nonholonomic deformations.[5] Such equation can be written in the form

$$\frac{\partial g_{ij}}{\partial\chi} = -2\widehat{R}_{ij}, \quad \frac{\partial h_{ab}}{\partial\chi} = -2\widehat{R}_{ab}, \tag{13}$$
$$\widehat{R}_{ia} = 0 \text{ and } \widehat{R}_{ai} = 0.$$

For simplicity, we shall analyze Ricci flows of families of ansatz for d–metrics parametrized in the form

$$\chi\mathbf{g} = e^{\psi(x^k)}dx^i \otimes dx^i + h_3(x^k,t,\chi)\mathbf{e}^3\otimes\mathbf{e}^3 + h_4(x^k,t,\chi)\mathbf{e}^4\otimes\mathbf{e}^4, \tag{14}$$
$$\mathbf{e}^3 = dt + w_i(x^k,t,\chi)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k,t,\chi)dx^i.$$

The system of equations (13) for $\widehat{R}_{ij} = 0$ and $\Upsilon_2(x^k,v) = \lambda$ and existing one Killing symmetry, on vector $\partial/\partial y^4$ (the coefficients do not depend on variable y^4), with evolution only of the v–parts, transform into

$$\ddot{\psi} + \psi'' = 0, \tag{15}$$
$$\frac{\partial}{\partial\chi}h_3 = -\frac{h_3\phi^*}{h_4}, \frac{\partial}{\partial\chi}h_4 = -\frac{h_4\phi^*}{h_3}, \tag{16}$$
$$\beta w_i + \alpha_i = 0, \tag{17}$$
$$n_i^{**} + \gamma n_i^* = 0 \tag{18}$$

where

$$\phi(\chi) = \phi(x^k,t,\chi) = \ln\left|\frac{h_4^*}{\sqrt{|h_3 h_4|}}\right|, \tag{19}$$
$$\alpha_i = h_4^*\partial_i\phi, \ \beta = h_4^* \phi^*, \ \gamma = \left(\ln|h_4|^{3/2}/|h_3|\right)^*.$$

In the above formulas we wrote the partial derivatives in the form $a^\bullet = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial t$ and we shall use $\partial_\chi a = \partial a/\partial\chi$. The conditions of zero torsion, i. e. constraints (6), are written in the form

$$w_i^* = \mathbf{e}_i \ln|h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k, \ n_i^* = 0, \ \partial_i n_k = \partial_k n_i. \tag{20}$$

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[5] In next section, we derive such equations for stochastic Ricci flows and related Perelman's functionals.

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We have to impose additionally (20) if we wont to consider nonholonomic Ricci flows determined by equations (15)–(18) in a form when the configurations for the Levi–Civita connections $\nabla(\chi)$ are extracted.

If $h_4^*(\chi) \neq 0; \Upsilon_2 = \lambda \neq 0$, we get $\phi^*(\chi) \neq 0$ and we can generate families of exact solutions of (5), with diagonal nontrivial source $\Upsilon_\delta^\alpha = diag[\lambda, \lambda, 0, 0]$, if

$$h_4^*(\chi) = 2h_3(\chi)h_4(\chi)\lambda/\phi^*(\chi). \tag{21}$$

We conclude that the family of ansatz for d–metrics (14) subjected to the conditions (15)–(21) define Ricci flow evolutions on a real parameter χ of a class of generic off–diagonal solutions determining correspondingly (non) holonomic Einstein manifolds. They depend explicitly on the type of families of generating functions $\phi(\chi)$. If such functions are random ones subjected to the conditions to solve certain stochastic/diffusion equations, we can say that we generated stochastic Einstein equations (for details, see our partner work [1] where there are also analyzed classes of solutions of Einstein equations with $h_4^* = 0$, or $h_3^* = 0$; the length of this paper does not allow us to analyze the Ricci flow evolution of such more special classes of Einstein manifolds) evolving, in general, randomly, on parameter χ.

3 Nonholonomic Diffusion and Self–Organized Criticality

We studied various examples with physically important solutions (black holes/ellipsoid, wormholes, pp– and/or solitonic waves etc) evolving under nonholonomic Ricci flows [49, 50, 28, 29]. Those solutions were with sure coefficients and sources. To our knowledge, it was not yet analyzed the evolution of, in general, non–Riemann geometries when certain coefficients of metrics and connections are random ones, i.e. the Ricci flow theory is with stochastic evolution.

In this work, we shall develop such a theory of stochastic Ricci flows, for simplicity, for families of metrics (14), which allows us to generate solutions in explicit form and to prove the phenomena of self–organizing criticality of gravitational fields for various stationary and stochastic Ricci flow evolution. The results from papers [2, 3, 4, 5] are crucial in proving the existence of unique and positive solutions. It should be emphasized here that the formalism of nonholonomic distributions with associated N–connections is a very important geometric tool for connecting the theory of Ricci flows to nonlinear diffusion and gravity theories and possible applications.

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3.1 Stochastic diffusion equations on nonholonomic manifolds

We studied some examples of nonlinear stochastic diffusion equations on nonholonomic $\mathbf{V}$ in the partner work [1], for the so–called N–adapted $(\mathbf{L}, \mathbf{A})$–diffusion and stochastic solutions of Einstein equations. It was constructed the corresponding Laplace–Beltrami operator for the canonical d–connection $\widehat{\mathbf{D}}$. We can generate stochastic d–metrics of type (14) if we take random generating functions $\phi(\chi)$ (19), with stochastic evolution parameter χ. Using (21), various families of exact solutions of Einstein equations can be defined. An important mathematical/physical problem is to state certain general conditions when the Ricci flow evolution equation have unique and positive solutions.

Let us introduce of nonholonomic geometric and N–adapted stochastic calculus framework. We consider an open bounder domain $\mathcal{U} \subset \mathbf{V}$,[6] with the spatial dim $\mathbf{V} \leq 3$ with smooth boundary $\partial \mathcal{U}$, when the Laplace–Beltrami operator $\widehat{\triangle}$ is completely defined by data $\left(\mathbf{N}, \mathbf{g}, \widehat{\mathbf{D}} \right)$. Out stochastic nonholonomic Ricci flows will be modelled using a nonlinear evolution equation

$$
\begin{aligned}
\delta U(\chi) - \widehat{\triangle}\Psi(U(\chi))\delta\chi &\ni \sigma(U(\chi))\delta\mathcal{W}(\chi), \text{ on } (0,\infty) \times \mathcal{U}, \\
\Psi(U(\chi)) &\ni 0, \text{ on } (0,\infty) \times \partial\mathcal{U}, \\
U(0,u) &= u \text{ on } \mathcal{U}.
\end{aligned}
\tag{22}
$$

In the above formulas, $\delta\mathcal{W}(\chi)$ is a Wiener process, the initial datum u is given for "rolling" the stochastic process on nonholonomic curved manifold $\mathbf{V}$, locally on carts of a covering atlas and we can consider any maximal monotone (possible multivalued) graph with polynomial growth of "coercive" function $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$. It is also possible to introduce a correspondingly parametrized random forcing term

$$
\sigma(U)dW = \sum_{k=1}^{\infty} \nu_k U \langle l, e_k \rangle_2 e_k
\tag{23}
$$

for any $l \in L^2(\mathcal{U})$, where $\langle \cdot, \cdot_k \rangle_2$ and e_k are respectively the scalar product and an orthonormal basis in $L^2(\mathcal{U})$ (which is determined, in our case, for a given d–metric structure on $\mathbf{V}$), when ν_k is a sequence of positive number and the set $\beta_k = \langle l, e_k \rangle$ can be associated to a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\chi\}_{\chi \geq 0}, \mathbb{P})$.

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[6]we suppose that our nonholonomic manifold is covered by such open regions

The physical meaning of equations (22) depends on the type of models we are going to elaborate. For instance, such an equation described the dynamics of flows in porous media; for more general assumptions, it models phase transitions with melting and solidification processes in the presence of a random force, i. e. term. Our proposal [1] was to introduce random generating functions for solutions in gravity (with the possibility to include noise for matter energy–momentum) considering vacuum non–vacuum spacetime configurations as a complex "ether" media (following Mach ideas on inertial forces and gravity) with possible nonholonomic structure, singularities and diffusion of gravitational and matter fields.

In this work, we show that gravitational fields (curved spacetime) may self–organize itself due to stochastic gravitational effects and/or under non-holonomic stochastic Ricci flow evolution. It is possible to formulate stochastic analogs of gravitational Ricci flow evolution equations (a geometric nonlinear diffusion driven by with stochastic components of the Ricci tensor, with certain limits to nonholonomic versions Laplace–Beltrami operators).

Our goal is to prove (using a synthesis of methods and results from [2, 3, 4, 5] and Perelman's functional approach [43, 44, 45] generalized for non-holonomic geometries [6, 7]) that a class of stochastic evolution/gravitational field equations can uniquely solved in a form satisfying positive conditions. There will be used sure evolution analogs of equations (22) for a function $\widehat{f}(x^1, x^2, v, \chi)$ which is modelled by

$$\frac{\partial \widehat{U}}{\partial \chi} = -\widehat{\Delta}\widehat{U} + \left|\widehat{\mathbf{D}}\widehat{U}\right|^2 - R - S, \qquad (24)$$

where $R + S$ is the scalar curvature of $\widehat{\mathbf{D}}$. Considering random generating functions we induce a gravitational random forcing term in formula (23). We say that Ψ is such way defined that the terms $-\widehat{\Delta}\widehat{U} + \left|\widehat{\mathbf{D}}\widehat{U}\right|^2$ are analogs of $\widehat{\Delta}\Psi$ in (22) when a unique solution of this equation is related to a unique solution of (24) in direct form or computing certain expectation values. We shall also consider unique stochastic solution for any well defined functions $\widehat{f}(\widehat{U}) = \widehat{f}(u) = \widehat{f}(x^1, x^2, v, \chi)$.

3.2 Existence of unique and positive solutions; physical setting for gravity and Lagrange–Finsler spaces

On flat three dimensional real spaces, the existence problem for stochastic equations (22) with additive and multiplicative noise was studied in Ref. [3] and, with generalized conditions, in [2, 4]. We shall search for general

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conditions of existence of equations (22) and (24) on $\mathcal{U} \subset \mathbf{V}$ under such assumptions:

Hypothesis 3.1 *1. The partition with $\mathcal{U} \subset \mathbf{V}$ and localization of operators $\widehat{\Delta}$ (the domain of this operator is $H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$), see below Notation 3.1, and $\widehat{\mathbf{D}}$ are such way parametrized via nonholonomic distributions that the coercive function Ψ is a maximal monotone multivalent function from $\mathbb{R}$ into $\mathbb{R}$, when $0 \in \Psi(0)$;*

2. $\exists \, C > 0$ and $a \geq 0$ when $\forall \in \mathbb{R}$ we can write $\sup\{|\theta| : \theta \in \Psi(r)\} \leq C\left(1 + |r|^a\right)$;

3. it is possible to fix the nonholonomic distributions, the canonical d–connection $\widehat{\mathbf{D}}$ and sequence ν_k in such a form that locally $\sum_{k=1}^{\infty} \nu_k^2 \lambda_k^2 < +\infty$, for λ_k being the eigenvalues of the Laplace–Beltrami d–operator $-\widehat{\Delta}$ on $\mathcal{U} \subset \mathbf{V}$ with Dirichle boundary conditions on $\partial\mathcal{U}$.

Roughly speaking, the conditions of above hypothesis are defined to "roll" on $\mathcal{U}$, in N–adapted form, for the canonical d–connection $\widehat{\mathbf{D}}$ and $\widehat{\Delta}$, the assumptions from Hypothesis 1.1 in Ref. [2]. Similarly, the equations (22) and further developments in this sections are "nonholonomic modifications" of the results on nonlinear diffusion from the mentioned papers [2, 3, 4] but related to stochastic Ricci flows via profs of existence for (24).[7]

Notation 3.1 *1. We have $\sigma(u) \in L_2\left(L^2(\mathcal{U}), H^{-1}(\mathcal{U})\right)$ with corresponding spaces/ conditions for all Hilbert–Schmidt d–operators from $L^2(\mathcal{U})$ into $H^{-1}(\mathcal{U})$, when there is the Lipschitz continuity from $H^{-1}(\mathcal{U})$ into $L_2\left(L^2(\mathcal{U}), H^{-1}(\mathcal{U})\right)$.*

2. By $L^p(\mathcal{U}), p \geq 1$, we denote the space of p–integrable functions with norm $|\cdot|_p$. The spaces $H^k(\mathcal{U}) \subset L^2(\mathcal{U})$, (k = 1, 2), are the standard Sobolev spaces on $\mathcal{U}$. We consider that $H_0^1(\mathcal{U})$ is the subspace of $H^1(\mathcal{U})$ with zero trace on the boundary.

3. Denoting $\mathcal{H}$ as a Hilbert space, for $p, q \in [1, +\infty]$, we write $L_W\left[\left(0, {}^T\chi\right); L^p(\Omega; \mathcal{H})\right]$ for the space of all q–integrable processes z :

---

[7]We recommend readers to consult those papers on nonlinear diffusion and self–organized criticality where basic concepts and veri important results are stated following a rigorous mathematical formalism. The limits of this work does not allow us to repeat and develop in a detailed form those results for h– and v–splitting, with N–adapted constructions, on $\mathcal{U} \subset \mathbf{V}$.

$[0,\ ^{T}\chi] \to L^{p}(\Omega; \mathcal{H})$ *are adapted to the filtration* $\{\mathcal{F}_{\chi}\}_{\chi \geq 0}$. *It is also used* $C_{W}\left[\left(0,\ ^{T}\chi\right); L^{2}(\Omega; \mathcal{H})\right]$ *for the space of all* $\mathcal{H}$*–valued N–adapted processes being mean square continuous.* $L(\mathcal{H})$ *denotes the space of bounded linear operators equipped with the above introduced norm.*

4. *We model* $\mathcal{H}$ *as a distribution space* $\mathcal{H} = H^{-1}(\mathcal{U}) = \left(H_{0}^{1}(\mathcal{U})\right)'$ *provided with a N–adapted scalar product and norm defined by d–metric* $\mathbf{g}$ *and canonical Laplace–Beltrami d–operator,* $\mathbf{A} = -\widehat{\Delta}$,

$$\langle\ ^{1}f,\ ^{2}f\rangle = \int_{\mathcal{U}} \mathbf{A}^{-1}\ ^{1}f(\xi)\ ^{2}f(\xi)\sqrt{|\mathbf{g}(\xi)|}\delta\xi,\ \text{for}\ |f|_{-1} = \sqrt{\langle\ f,\ f\rangle}.$$

Under above assumptions, on any $\mathcal{U} \subset \mathbf{V}$ and for a parameter χ, we can use the main result from [2] that if $u \in L^{p}(\mathcal{U}), p \geq \max\{2a, 4\}$, then there is a unique strong solution to equation (22) (any such solution is nonnegative if the initial data u are also nonnegative).

In our nonholonomic setting for gravity and flows of geometries, we can solve different physical problems: We do not model usual porous media, but a gravitational "ether" interacting as a solution of Einstein equations and/or following a scenarios of Ricci flow stochastic evolution. For instance, there are some important examples of modeling for gravitational configurations by nonholonomic diffusion equations of type (22):

Example 3.1 *1. For certain constants* $^{0}\chi, \rho,\ ^{1}\alpha,\ ^{2}\alpha \in (0, +\infty)$, *we can model a gravitational thermo–field theory with heat conduction, or phase transitions in spacetime "porous foam" if*

$$\Psi(\chi) = \left\{ \begin{array}{l} ^{1}\alpha(\chi -\ ^{0}\chi),\ \text{for}\ \chi <\ ^{0}\chi; \\ [0, \rho],\ \text{for}\ \chi =\ ^{0}\chi; \\ ^{2}\alpha(\chi -\ ^{0}\chi) + \rho,\ \text{for}\ \chi >\ ^{0}\chi. \end{array} \right. \left.\rule{0pt}{36pt}\right|$$

2. *In a spacetime with black holes and singularities, it is important to consider the nonlinear singular nonholonomic diffusion equation*

$$\delta U(\chi) - \rho div\left[\ ^{0}\delta\left(U(\chi)\right)\widehat{\mathbf{D}}U(\chi)\right]d\chi = \sigma\left(U(\chi)\right)\delta\mathcal{W}(\chi),$$

for N–adapted divergence div, where $^{0}\delta$ *is the Dirac measure concentrated at the origin, which is possible for* $\Psi(\chi) = \left\{ \begin{array}{l} \rho\chi/|\chi|,\ \text{if}\ \chi \neq 0; \\ [-1, 1],\ \text{if}\ \chi = 0. \end{array} \right. \left.\rule{0pt}{20pt}\right|$

3. *Choosing* $\Psi(\chi) = |\chi|^{\alpha}sign\chi$ *with* $0 < \alpha \leq 1$, *for a local diffusion problem with free boundary and a random forcing term proportional to*

$U(\chi) - \,^{c}\chi$, *when the value* $\,^{c}\chi$ *determine the critical point of diffusion process* $U(\chi)$, *we get the a particular nonholonomic diffusion equation*

$$\delta U(\chi) - \widehat{\Delta}\left(Hev(\chi) + \varkappa\right)\left[U(\chi) - \,^{c}u \,\right] d\chi = \sigma\left(U(\chi) - \,^{c}u \,\right)\delta\mathcal{W}(\chi).$$

In this formula, it is used the Heavside step function $Hev(\chi) = \{0, \text{ if } \chi < 0; [0,1], \text{ if } \chi = 0; 1, \text{ if } \chi > 0\}$. *Certain physical models for self–organized criticality and their rigorous mathematical study are given in Refs. [37, 36, 2]. For spacetime evolution, such criticality can be related with behavior of certain effective physical constants and/or diffusion gravitational phase transitions subjected to nonholonomic constraints. One might consider scenarios when the supercritical region* $\{U(\chi) > \,^{c}u\}$ *is absorbed asymptotically during evolution by the critical one* $\{U(\chi) = \,^{c}u\}$. *Such models seem to have applications in modern quantum gravity (as alternative to Horava–Lifshitz phase transitions) and or in cosmology driven by Ricci flow models.*

For the above given examples, we can not apply the general existence theory of infinite dimensional stochastic equations in Hilbert space (enabled with nonlinear maximal monotone operators). We have to elaborate a N–adapted approach.

Definition 3.1 *We call a solution of (22) any* $\mathcal{H}$*–valued continuous* $\mathcal{F}_{\chi}$*–adapted and N–adapted process* $U(\chi) = U(\chi, z)$, *for* $z \in \mathcal{H}$, *on* $[0, \,^{T}\chi]$ *if* $U \in L^{p}\left(\Omega \times (0, \,^{T}\chi) \times \mathcal{U}\right) \cap L^{2}\left(0, \,^{T}\chi; L^{2}(\Omega; \mathcal{H})\right), p \geq a$, *and* $\exists \eta \in L^{p/a}\left(\Omega \times (0, \,^{T}\chi) \times \mathcal{U}\right)$ *such that* $\mathbb{P}$*-a.s.*

$$\langle U(\chi, z), \ e_{j}\rangle_{2} = \langle z, \ e_{j}\rangle_{2} +$$

$$\int_{0}^{\chi}\int_{\mathcal{U}} \eta(s, \xi)\widehat{\Delta}e_{j}(\xi)\sqrt{|\mathbf{g}(\xi)|}\delta\xi ds + \sum_{k=1}^{\infty}\nu_{k}\int_{0}^{\chi}\langle U(s, z) \ e_{k}, e_{j}\rangle_{2} d\beta_{k}(s),$$

$\forall j \in \mathbb{N}$ *and* $\eta \in \Psi(U)$ *a.e. in* $\Omega \times (0, \,^{T}\chi) \times \mathcal{U}$.

Finally, we formulate a N–adapted generalization of the existence theorem (Main Result of [2]):

Theorem 3.1 *For each* $z \in L^{p}(\mathcal{U}), p \geq \max\{2a, 4\}$ *and conditions of Hypothesis 1, there is a unique solution* $U \in L_{W}^{\infty}\left[(0, \,^{T}\chi); L^{p}(\Omega; \mathcal{U})\right]$, *see Notation 3.1, to (22). If additionally* z *is nonnegative a.e. in* $\mathcal{U}$ *then* $\mathbb{P}$*-a.s.* $U(\chi, z)(\xi) \geq 0$, *for a.e.* $(\chi, \xi) \in (0, \infty) \times \mathcal{U}$.

Proof is similar to that provided for flat spaces. We have to "roll" on atlas carts and dub the constructions for h– and v–components and using the Laplace–Beltrami operator for $\widehat{\Delta}$ determined by the canonical d–connection $\widehat{\mathbf{D}}$. We omit such technical results. The most important consequence of this Theorem is that the existence of a unique solution for such diffusion equations can be related to stochastic Ricci flows. This allows us to derive in a unique form the evolution equations and related (stochastic) fundamental functionals.

4 Stochastic Nonholonomic Ricci Flows

For a general random generating function $\phi(\chi)$ introduced into a d–metric (14), it is not clear how to define the Perelman functionals and derive the Hamilton evolution equations for stochastic Ricci flows. The goal of this section is to sketch in brief a self–consistent stochastic version of Hamilton–Perelman theory for nonlinear diffusion of metric coefficients when the assumptions of Hypothesis 1 and conditions of Theorem 3.1 are satisfied for certain generating/normalizing functions.

4.1 A stochastic modification of Perelman's functionals

The Perelman's functionals were introduced for Ricci flows of Riemannian metrics and Levi–Civita connection [43, 44, 45] are written in the form

$$
{}_{|}\mathcal{F}(\mathbf{g}, f) = \int_{\mathbf{V}} \left({}_{|}R + |\nabla f|^2 \right) e^{-f} \; dV, \tag{25}
$$

$$
{}_{|}\mathcal{W}(\mathbf{g}, f, \tau) = \int_{\mathbf{V}} \left[\tau \left({}_{|}R + |\nabla f| \right)^2 + f - 2n \right] \mu \; dV,
$$

where $dV(\xi) = \sqrt{|\mathbf{g}(\xi)|}\delta\xi$ is the volume form, integration is taken over compact $\mathbf{V}$ and ${}_{|}R$ is the scalar curvature computed for ∇. For a real evolution parameter $\tau > 0$, it is considered $\int_{\mathbf{V}} \mu dV = 1$ when $\mu = (4\pi\tau)^{-n} e^{-f}$. The functional approach can be redefined for N–anholonomic manifolds with stochastic generating functions (in this section, we shall follow the N–adapted geometric formalism elaborated in Refs. [6, 8]):

Claim 4.1 –*Definition: For nonholonomic manifolds of even dimension $2n$ with stochastically generated geometric objects, the stochastic Perelman's*

functionals for the canonical d–connection $\widehat{\mathbf{D}}$ *are defined*

$$\widehat{\mathcal{F}}(\mathbf{g}, \widehat{f}) \;=\; \int_{\mathbf{V}} \left(R + S + \left|\widehat{\mathbf{D}}\widehat{f}\right|^2 \right) e^{-\widehat{f}} \; dV, \tag{26}$$

$$\widehat{\mathcal{W}}(\mathbf{g}, \widehat{f}, \tau) \;=\; \int_{\mathbf{V}} \left[\widehat{\tau} \left(R + S + \left|{}^h D\widehat{f}\right| + \left|{}^v D\widehat{f}\right| \right)^2 + \widehat{f} - 2n \right] \widehat{\mu} \; dV, \tag{27}$$

where dV is the volume form of ${}^L\mathbf{g}$, R and S are respectively the h- and v–components of the curvature scalar of $\widehat{\mathbf{D}}$, for $\widehat{\mathbf{D}}_\alpha = (D_i, D_a)$, or $\widehat{\mathbf{D}} = ({}^h D, \; {}^v D)$, $\left|\widehat{\mathbf{D}}\widehat{f}\right|^2 = \left|{}^h D\widehat{f}\right|^2 + \left|{}^v D\widehat{f}\right|^2$, and $\widehat{f}$ satisfies $\int_{\mathbf{V}} \widehat{\mu} dV = 1$ for $\widehat{\mu} = (4\pi\tau)^{-n} e^{-\widehat{f}}$ and $\tau > 0$.

Proof. The formulas (25) are redefined for some $\widehat{f}$ and f (which can be a non–explicit relation between random functions etc and with gravitational "noise" induced by stochastic N–adapted $R + S$) when

$$\left({}_{|}R + |\nabla f|^2 \right) e^{-f} = \left(R + S + \left|{}^h D\widehat{f}\right|^2 + \left|{}^v D\widehat{f}\right|^2 \right) e^{-\widehat{f}} + q.$$

Re–scaling the evolution parameter τ (it is similar to the considered above χ), $\tau \to \widehat{\tau}$, we have

$$\left[\tau \left({}_{|}R + |\nabla f| \right)^2 + f - 2n \right) \right] \mu = \left[\widehat{\tau} \left(R + S + \left|{}^h D\widehat{f}\right| + \left|{}^v D\widehat{f}\right| \right)^2 + \widehat{f} - 2n \right] \widehat{\mu} + q_1$$

for some q and q_1 for which $\int_{\mathbf{V}} q dV = 0$ and $\int_{\mathbf{V}} q_1 dV = 0$. □

The geometric objects defining functionals $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{W}}$ are with some components computed as expectation values (see details in [1], for instance, formulas (12) - (14) when the probability density function subjected to Focker–Plank conditions, is used for computing such values). We consider the h–variation ${}^h\delta g_{ij} = v_{ij}$, the v–variation ${}^v\delta g_{ab} = v_{ab}$, and ${}^h\delta\widehat{f} = {}^h f, {}^v\delta\widehat{f} = {}^v f$

An explicit calculus for the first N–adapted variations of (26), see similar details in [6, 8], distinguished into sure and random components, can be represented in the form

$$\delta\widehat{\mathcal{F}}(v_{ij}, v_{ab}, \; {}^h f, \; {}^v f) = \tag{28}$$

$$\int_{\mathbf{V}} \{ [-v_{ij}(R_{ij} + D_i D_j \widehat{f}) + (\frac{{}^h v}{2} - \; {}^h f)\left(2 \; {}^h \Delta\widehat{f} - |\; {}^h D \; \widehat{f}| \right) + R]$$

$$+ [-v_{ab}(R_{ab} + D_a D_b \widehat{f}) + (\frac{{}^v v}{2} - \; {}^v f)\left(2 \; {}^v \Delta\widehat{f} - |\; {}^v D \; \widehat{f}| \right) + S] \} e^{-\widehat{f}} dV,$$

where $^h\Delta = D_i D^i$ and $^v\Delta = D_a D^a$, $\widehat{\Delta} = {}^h\Delta + {}^v\Delta$, and $^hv = g^{ij}v_{ij}$, $^vv = h^{ab}v_{ab}$.

4.2   Main Theorems for stochastic Ricci flow equations

We shall prove that stochastic equations of type (13) can be derived from the Perelman's N–adapted functionals (26) and (27) (for simplicity, we shall not consider the normalized term and put $\lambda = 0$).

Definition 4.1 *A metric* **g** *generated by a stochastic generating function $\phi(\chi)$ is called a (nonholonomic) stochastic breather if for some $\chi_1 < \chi_2$ and $\alpha > 0$ the metrics α **g**(χ_1) and α **g**(χ_2) differ only by a N–adapted diffeomorphism. The cases $\alpha =, <, > 1$ define correspondingly the steady, shrinking and expanding breathers (there are configurations when, for instance, the h–component of metric is steady but the v–component is shrinking).*

Clearly, the breather properties, in sure and stochastic variables, depend on the type of connections which are used for definition of Ricci flows. We can elaborate a unique nonlinear diffusion scenarios for $\widehat{\mathbf{D}}$ for the assumptions of Hypothesis 1.

Following a N–adapted variational calculus for $\widehat{\mathcal{F}}(\mathbf{g}, \widehat{f})$, with formula (28), Laplacian $\widehat{\Delta}$ and h- and v–components of the Ricci tensor, $\widehat{R}_{ij}$ and $\widehat{S}_{ij}$, and considering parameter $\tau(\chi)$, $\partial\tau/\partial\chi = -1$, we formulate

Theorem 4.1 *The stochastic nonholonomic Ricci flows can be parametrized by corresponding nonholonomic distributions and characterized by evolution equations*

$$\frac{\partial g_{ij}}{\partial \chi} = -2\widehat{R}_{ij}, \ \frac{\partial h_{ab}}{\partial \chi} = -2\widehat{R}_{ab},$$

$$\frac{\partial \widehat{f}}{\partial \chi} = -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathbf{D}}\widehat{f}\right|^2 - R - S \qquad (29)$$

and the property that

$$\frac{\partial}{\partial \chi}\widehat{\mathcal{F}}(\mathbf{g}(\chi), \widehat{f}(\chi)) = 2\int_{\mathbf{V}} \left[|\widehat{R}_{ij} + D_i D_j \widehat{f}|^2 + |\widehat{R}_{ab} + D_a D_b \widehat{f}|^2\right] e^{-\widehat{f}} dV,$$

$\int_{\mathbf{V}} e^{-\widehat{f}} dV$ *is constant and the geometric objects $\widehat{\mathbf{D}}, \widehat{\Delta}, \widehat{R}_{ij}, \widehat{R}_{ab}, R$ and S are induced from random generating functions in* **g**(χ).

Proof. We sketch the idea of such a proof: We should follow G. Perelman [43] constructions (details are given for the connection ∇ in the Proposition 1.5.3 of [46]) but for the canonical d–connection $\widehat{\mathbf{D}}$). The random components of metrics and connections are included as mathematical expectations. The most important task is to prove that for stochastic generating functions the equation (29) has a unique solution for well defined conditions. Really, this equation is equivalent to (24) for with a stochastic system (22) can be associated. The existence of a unique solution follows from Theorem 3.1. □

We can also derive stochastic evolution equations, following stochastic N–adapted modifications of Proposition 1.5.8 in [46] containing the details of the original result from [43]:

Theorem 4.2 *If a d–metric* $\mathbf{g}(\chi)$ *determined by stochastic generating function* $\phi(\chi)$ *(19) and functions* $\widehat{f}(\chi)$ *and* $\widehat{\tau}(\chi)$ *evolve stochastically following the equations (a stochastic nonholonomic generalization of Hamilton's equations)*

$$\frac{\partial g_{ij}}{\partial \chi} = -2\widehat{R}_{ij}, \ \frac{\partial h_{ab}}{\partial \chi} = -2\widehat{R}_{ab},$$

$$\frac{\partial \widehat{f}}{\partial \chi} = -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathbf{D}}\widehat{f}\right|^2 - R - S + \frac{n}{\widehat{\tau}}, \tag{30}$$

$$\frac{\partial \widehat{\tau}}{\partial \chi} = -1 \tag{31}$$

and the property that that the stochastic Ricci flow evolution is constrained to satisfy, for $\int_{\mathbf{V}}(4\pi\widehat{\tau})^{-n}e^{-\widehat{f}}dV = const$, *the conditions*

$$\frac{\partial}{\partial \chi}\widehat{\mathcal{W}}(\ \mathbf{g}(\chi),\widehat{f}(\chi),\widehat{\tau}(\chi)) =$$
$$2\int_{\mathbf{V}} \widehat{\tau}[|\widehat{R}_{ij} + D_iD_j\widehat{f} - \frac{1}{2\widehat{\tau}}g_{ij}|^2 + |\widehat{R}_{ab} + D_aD_b\widehat{f} - \frac{1}{2\widehat{\tau}}g_{ab}|^2](4\pi\widehat{\tau})^{-n}e^{-\widehat{f}}dV.$$

The equation (30) is similar to (24) and can be related to a stochastic system of type (22). The condition (31) states a relation between a Ricci flow evolution parameter $\widehat{\tau}$ and stochastic evolution parameter χ.

4.3 Statistical analogy for stochastic Ricci flows

The Ricci flow theory has, in its turn, a very interesting application in the theory of stochastic equations and diffusion. It allows us to characterize

additionally such processes via associated statistical functionals, entropy and thermodynamical values. G. Perelman emphasized [43] that the functional $_{|}\mathcal{W}$ is in a sense analogous to minus entropy. In this section, we prove that such a property exists also for nonholonomic stochastic Ricci flows and that we can provide a statistical model for nonlinear diffusion processes.[8]

We consider a evolution of stochastic geometric systems described by some metrics $\mathbf{g}(\widehat{\tau})$, N–connections $N_i^a(\widehat{\tau})$ and related canonical d–connections $\widehat{\mathbf{D}}(\widehat{\tau})$ when the conditions of Theorem 4.2 are satisfied. It follows:

Theorem 4.3 *Any family of stochastically Ricci flow evolving nonholonomic geometries, and solutions of Einstein equations, with nonlinear diffusion data derived from Hypothesis 1 and conditions of Theorem 3.1 is characterized by thermodynamic values*

$$\langle \widehat{E} \rangle = -\widehat{\tau}^2 \int_{\mathbf{V}} \left(R + S + \left| {}^h D \widehat{f} \right|^2 + \left| {}^v D \widehat{f} \right|^2 - \frac{n}{\widehat{\tau}} \right) \widehat{\mu} \ dV, \tag{32}$$

$$\widehat{S} = - \int_{\mathbf{V}} \left[\widehat{\tau} \left(R + S + \left| {}^h D \widehat{f} \right|^2 + \left| {}^v D \widehat{f} \right|^2 \right) + \widehat{f} - 2n \right] \widehat{\mu} \ dV,$$

$$\widehat{\sigma} = 2 \ \widehat{\tau}^4 \int_{\mathbf{V}} \left[|\widehat{R}_{ij} + D_i D_j \widehat{f} - \frac{1}{2\widehat{\tau}} g_{ij}|^2 + |\widehat{R}_{ab} + D_a D_b \widehat{f} - \frac{1}{2\widehat{\tau}} g_{ab}|^2 \right] \widehat{\mu} \ dV.$$

Proof. It follows from a straightforward computation for $\widehat{Z} = \exp\{\int_{\mathbf{V}} [-\widehat{f} + n] \widehat{\mu} dV\}$ as in the original paper [43]. For nonholonomic stochastic processes, we have to N–adapt the constructions as in [6, 8]. The stochastic terms are included in formulas (32) via expected values of smooth coefficients satisfying the Fokker–Plank equation (or forward Kolmogorov equation). $\square$

Any N–adapted stochastic configuration determined by a canonical d–connection $\widehat{\mathbf{D}}$ is thermodynamically more (less, equivalent) convenient than a similar one defined by the Levi–Civita connection ∇ if $\widehat{S} < {}_{|}S$ ($\widehat{S} > {}_{|}S, \widehat{S} = {}_{|}S$). Similarly, a certain geometry can be more (less, equivalent) convenient than a stochastic one and related Ricci flow evolution models.

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[8]Let us remember some important concepts from statistical mechanics: The partition function $Z = \int \exp(-\beta E) d\omega(E)$ for the canonical ensemble at temperature $^T \beta^{-1}$ is defined by the measure taken to be the density of states $\omega(E)$. The thermodynamical values are computed in the form: the average energy, $\langle E \rangle = -\partial \log Z / \partial \ ^T \beta$, the entropy $S = {}^T \beta \langle E \rangle + \log Z$ and the fluctuation $\sigma = \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z / \partial \ ^T \beta^2$.

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